EFFECT OF THE ASYMMETRY OF A HEAT FLOW ON THE FORCE OF RESISTANCE OF A DROP IN A SLOW VISCOUS FLOW

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We show that if the energy required for the evaporation of a drop is supplied by molecular heat conduction in a vapor medium, then the calculation of corrections for the Stokes equation requires that we take into account the nonspherical nature of the vapor flow on the surface of the drop.

We study the quasistationary evaporation of a drop with internal heat release that is limited by an equation of heat balance. The temperature of the surface of the drop is assumed to be equal to the temperature of boiling. The transfer of energy from the drop to the vapor medium is realized by emission, molecular heat conduction, and convection. The temperature difference between the drop and the surface is assumed to be small in comparison to the temperature of the drop. The numbers Re and R, which are determined according to the velocity of the slow vapor flow and the velocity of the vapor near the surface of the drop, satisfy the conditions $\mathrm{Re} \ll R \ll 1$. The slow vapor results in asymmetry in the distribution of the temperature field in the region of the drop and, consequently, in the distribution of the velocity of the vapor on its surface.

The vapor motion around the drop is described by the Navier—Stokes equations and equations of continuity which, as is known [1], lead to the equation for the stream function

$$D^4\psi = \frac{\mathrm{Re}}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\cdot\frac{\partial}{\partial r} - \frac{\partial\psi}{\partial r}\cdot\frac{\partial}{\partial\theta} + 2\operatorname{ctg}\theta\frac{\partial\psi}{\partial r} - \frac{2}{r}\cdot\frac{\partial\psi}{\partial\theta}\right)D^2\psi,$$

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\cdot\frac{\partial}{\partial\theta}\cdot\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta},\quad \mathrm{Re} = \frac{ua}{\nu},\quad v_r = \frac{1}{r^2\sin\theta}\cdot\frac{\partial\psi}{\partial\theta},\quad v_\theta = -\frac{1}{r\sin\theta}\cdot\frac{\partial\psi}{\partial r}. \tag{1}$$

Here ψ and r are dimensionless stream functions and the distance to the center of the drop; v_r and v_θ are components of the dimensionless velocity in the spherical coordinate system with the polar axis along the direction **u** of the velocity of the flow at infinity; a is the radius of the drop; ν is the kinematic viscosity. We must replace r, v_r, v_θ, ψ by ar, uv_r, uv_θ, $ua^2\psi$ to transform to the common dimensional quantities.

The boundary conditions are

$$v_r = \frac{1}{\sin\theta}\cdot\frac{\partial\psi}{\partial\theta} = w(\theta),\quad v_\theta = \frac{\partial\psi}{\partial r} = 0 \text{ for } r = 1,$$
$$\psi \to \frac{1}{2}r^2\sin^2\theta \text{ as } r\to\infty. \tag{2}$$

Here $w(\theta)$ is the dimensionless radial component of the vapor velocity on the surface of the drop.

We seek the solution of Eq. (1) with boundary conditions (2) as

$$\psi = \psi_0 + \psi_1 + \psi_s. \tag{3}$$

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Here ψ_0 is the stream function of the potential flow that satisfies the equation $D^2\psi_0 = 0$ and the boundary conditions

$$\frac{\partial\psi_0}{\partial\theta} = w \sin\theta \quad \text{for} \quad r = 1, \ \frac{\partial\psi_0}{\partial r} \to 0 \quad \text{as} \quad r \to \infty.$$

If we expand the velocity w in series in Legendre polynomials and limit ourselves to the first two terms, which assumes the presence of the small parameter determined below, then the stream function will take the form

$$\psi_0 = -w_0 \cos\theta + \frac{w_1}{2r} \sin^2\theta.$$

The stream function ψ_S in Eq. (3) that describes the flow of the solid sphere in a Stokes approximation takes the form

$$\psi_s = \frac{1}{4}\left(2r^2 - 3r + \frac{1}{r}\right)\sin^2\theta.$$

The superposition of the stream function $\psi_0 + \psi_S$ does not satisfy the second boundary condition of (2). If we seek a stream function with accuracy up to terms of order R inclusive, then we obtain the following equation to find ψ_1:

$$D^4\psi_1 = -\frac{9R}{2r^4}\sin^2\theta \quad (R = w_0 \mathrm{Re}) \tag{4}$$

with boundary conditions

$$\frac{\partial\psi_1}{\partial\theta} = 0, \quad \frac{\partial\psi_1}{\partial r} = \frac{w_1}{2}\sin^2\theta \quad \text{for} \quad r = 1,$$

$$\frac{1}{r^2}\psi_1 \to 0 \quad \text{as} \quad r \to \infty. \tag{5}$$

Equation (4) follows from (1) if we replace ψ with ψ_0 and $D^2\psi$ with $D^2(\psi_0 + \psi_S) = D^2\psi_S$ in the convective term before the operator $D^2\psi$.

The solution of Eq. (4) with boundary conditions (5) takes the form

$$\psi_1 = -\frac{9R}{32}\left(r - 2 + \frac{1}{r}\right)\sin^2\theta + \frac{w_1}{4}\left(r - \frac{1}{r}\right)\sin^2\theta. \tag{6}$$

We note that the solution obtained corresponds to the partial computation of the convective terms in Eq. (1), which is justified in region r of order 1 if the expression $\mathrm{Re} \ll R \ll 1$ is realized.

The pressure in the vapor medium is determined from the Navier – Stokes equation, which is presented in dimensionless form for the convective terms given above:

$$\nabla(v_0^2 + \mathbf{v}_0 \cdot \mathbf{v}_s + p) - [\mathbf{v}_0 \times \mathrm{rot}\,\mathbf{v}_s] = -\frac{1}{\mathrm{Re}}\Delta(\mathbf{v}_s + \mathbf{v}_1).$$

Here $\mathbf{v}_0$, $\mathbf{v}_S$, and $\mathbf{v}_1$ are dimensionless velocity fields determined by the stream functions ψ_0, ψ_S, ψ_1; p is the dimensionless pressure which should be replaced by $\rho u^2 p$ (ρ is the density of the vapor) in transforming to the dimensional quantities.

From this equation the part of the pressure on the boundary of the drop that is essential for calculating the resistance force takes the following form with the expressions for ψ_0, ψ_S, and ψ_1 taken into account:

$$p = \left(-\frac{3}{2\mathrm{Re}} + \frac{3}{16}w_0 + \frac{1}{2\mathrm{Re}}w_1\right)\cos\theta + \dots.$$

The force acting on the drop is determined according to the known velocity field $\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_S + \mathbf{v}_1$ and the pressure. It is evident that the direction of the force coincides with the direction of the velocity of the slow flow $\mathbf{u}$.

Thus, the projection of the force on the direction $\bar{\mathbf{u}}$ is equal to [2]

$$F = 2\pi a^2 \rho u^2 \int_0^\pi \left[-v_r^2\cos\theta - p\cos\theta + \frac{2}{\mathrm{Re}} \cdot \frac{\partial v_r}{\partial r}\cos\theta - \frac{1}{\mathrm{Re}}\left(\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r}\right)\sin\theta\right]\sin\theta\, d\theta.$$

Thus, the equation for the force with the degree of accuracy under study takes the form

$$F = 6\pi\rho vau\left(1 - \frac{7}{24}R - \frac{1}{3}w_1\right). \qquad (7)$$

The first term of this equation agrees with the Stokes equation [2], which determines the resistance of the sphere in the viscous flow. The second term in (7) agrees with the results obtained in [3, 4].

We must calculate the heat flow on the surface of the drop to determine the coefficients w_0 and w_1 in the equation for the resistance force (7).

At the sufficiently high thermal conductivity and thermal diffusivity of the fluid inside the drop, we assume the temperature of its surface T_a to be the same throughout, and the influx of energy from the drop to its surface is easily determined in this case by the equation for the energy balance.

The boundary condition on the surface of the drop takes the form

$$\frac{1}{3}\rho' ka = -\frac{\varkappa}{a}\left(\frac{\partial T}{\partial r}\right)_a + \rho Lu(w_0 + w_1\cos\theta) + 4\varepsilon\sigma T_\infty^3(T_a - T_\infty). \qquad (8)$$

Here ρ' is the density of the drop; k is the intensity of the internal heat release per unit mass; $\varkappa$ is the thermal-conductivity coefficient of the vapor medium; L is the heat of the phase transfer per unit mass; ε is the effective degree of the blackness of the drop surface; σ is the Stefan–Boltzmann constant. The quantities referring to the surface of the drop are denoted by the index a, and those referring to the region distant from it are denoted by the index ∞.

We assume that the emission does not have a significant effect on the temperature in the region of the drop. This assumption is justified, as shown in [5], if we satisfy the conditions

$$l \gg \frac{a}{3}\sqrt{\frac{\varkappa_r}{\varkappa}} \text{ and } l \gg a,$$

where l is the mean free path of the emission, and $\varkappa_r$ is the coefficient of radiant thermal conductivity.

The temperature distribution in the region of the drop is determined by an equation of convective heat conductivity:

$$\Delta\xi = \frac{\mathrm{Pe}}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\cdot\frac{\partial\xi}{\partial r} - \frac{\partial\psi}{\partial r}\cdot\frac{\partial\xi}{\partial\theta}\right); \quad \xi = \frac{T - T_\infty}{T_a - T_\infty}, \qquad \mathrm{Pe} = \frac{ua}{\chi}.$$

Here χ is the molecular thermal conductivity. The boundary conditions are

$$\xi = 1 \text{ for } r = 1, \quad \xi \to 0 \text{ as } r \to \infty.$$

The solution of the equation can be obtained by the method of joint asymptotic expansions [6, 7].

The monomial internal expansion takes the form $\xi_* = 1/r$, which allows us to determine the monomial external expansion

$$\xi^* = \frac{1}{r}\exp\left[-\frac{\mathrm{Pe}}{2}r(1 - \cos\theta)\right].$$

The binomial internal expansion $\xi_* = (1/r) + \mathrm{Pe}\,\xi_1$ is determined by the equation

$$\Delta\xi_1 = -\frac{1}{r^4\sin\theta}\cdot\frac{\partial\psi}{\partial\theta},$$

$$\xi_1(1, \theta) = 0, \quad \xi_1 \to \frac{1}{2}(\cos\theta - 1) \text{ as } r \to \infty.$$

It is sufficient to choose $\psi = \psi_0 + \psi_S$ in calculating ξ_1, since the computation of ψ_1 results in terms of a higher order of smallness appearing in ξ_1.

Thus, the binomial internal expansion takes the form

$$\xi_* = \frac{1}{r} + \frac{P}{2}\left(\frac{1}{r} - \frac{1}{r^2}\right) + \frac{\mathrm{Pe}}{2}\left[\frac{1}{r} - 1 + \left(1 - \frac{3}{2r} + \frac{3}{4r^2} - \frac{1}{4r^3}\right)\cos\theta\right]; \quad P = w_0\,\mathrm{Pe}.$$

It follows from boundary condition (8) that

$$w_0 = \frac{\rho' ka}{3\rho Lu} - \frac{\varkappa(T_a - T_\infty)}{\rho a Lu}\left(1 - \frac{P}{2} + \frac{\mathrm{Pe}}{2} + \frac{4\varepsilon\sigma a T_a^3}{\varkappa}\right),$$

$$w_1 = \frac{3\varkappa(T_a - T_\infty)}{8\rho a Lu}\mathrm{Pe} = \frac{3c_p}{8L}(T_a - T_\infty).$$

Here c_p is the heat capacity per unit mass of the vapor medium at constant pressure.

In the absence of internal heat release the evaporation of the drop occurring at a temperature when the radiant energy transfer is negligibly small in comparison to the molecular transfer can be studied as a sphericosymmetric evaporation if $\mathrm{Pe} \ll 1$. However, in calculating the effect of the evaporation velocity on the resistance force, we must consider the nonspherical nature of the evaporation velocity of the drop. In this case both corrections for the Stokes equation will be single-order quantities in Eq. (7). This is also valid for $R < \mathrm{Re} < 1$. In the latter case the necessity arises of taking the Oseen term in Eq. (7) into account.

In [8] the resistance force and the evaporation velocity conditioned by the molecular thermal conductivity are numerically studied with the flow of the compressible gas and the variable physical properties taken into account. However, the final analytic equations for the resistance force and the Nusselt number are not presented in [8].

Another problem arises if the vapor velocity on the surface of the drop is not related to the molecular thermal conductivity in the medium surrounding the drop, as, for example, when the energy needed for evaporation acts as a result of the intense internal heat release or the radiant flow in the low-capture medium. In this case, we can disregard w_1 in comparison to w_0, and the model of the sphericosymmetric evaporating drop will be suitable for seeking the resistance force and the evaporation velocity for $P \sim R \lessdot 1$. We must show that this model is also applicable for an intensely evaporating drop [9] ($\mathrm{Pe} < 1$, $P \cdot \mathrm{Pe} < 1$) in connection with the exponential decrease of the heat flow that is in proportion to the increase of the evaporation velocity of the drop.

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